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ON DYNAMIC RESPONSE OF PRESTRESSED CYLINDRICAL SHELLS—GREEN'S TENSOR TECHNIQUE

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Abstract—The integral solution for the displacement vector of the mid-surface of a finite, simply supported thin circular cylindrical shell with initial biaxial stress subjected to arbitrary three dimensional time-dependent surface loadings has been derived in terms of the associated Green's tensor kernel based upon Flugge's equations. As direct applications, this paper presents theoretical analyses for (a) the dynamic response of a prestressed cylindrical shell due to a cyclic traveling axially-symmetric ring load and (b) the transient and steady-state response of a prestressed shell subjected to a cyclic moving point force. It is found that in both problems there exists theoretically an infinite number of load movement frequencies to resonantly excite each mode of vibration of the shell, either with or without initial stress. The general dynamic solution of prestressed shells together with the Green's tensor representation and the analytic solutions for the specific problems are also valid for Donnell's equations, if definition of some coefficients is properly adjusted.

NOTATION

a	radius of shell mid-surface
e	circular frequency of radial variation of ring load
h	shell thickness
l	shell length
m	number of axial half waves
n	number of circumferential waves
$q_1(x, \beta, t), q_2, q_3$	time dependent surface loads per unit surface area in x, y, z directions respectively
t	time
$\vec{d}(s, \beta, t) = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$	vibratory displacement vector of shell mid-surface excluding uniform deformations induced by uniform initial stress
x, y, z	cylindrical coordinates of mid-surface
x_1	center position of axial oscillation of loads
$A_{ij}\,(k, j = 1, 2, 3)$	linear partial differential operators
E	Young's modulus
E'	identity matrix of order 3
F	$Eh/(1-\nu^2)$
$G(x, \beta, t; x_0, \beta_0, t_0)$	Green's tensor
$H(t - t_0)$	Heaviside step function
J	Bessel function of the first kind
$K_{ij}(x, \beta, t; x_0, \beta_0, t_0)$	Green's tensor components
N_1^0, N_2^0	constant initial stress resultants in x, y directions respectively, positive for tension
P_0	amplitude of cyclic moving point force
Q_0	amplitude of intensity of oscillatory moving ring load
S	shell surface
X	amplitude of axial movement
α, β	dimensionless cylindrical coordinates, $\alpha = x/a$, $\beta = y/a$
α_1, α_2	dimensionless initial stress parameters, $\alpha_1 = N_1^0/F$, $\alpha_2 = N_2^0/F$

δ	Dirac distribution
δ_{ij}	Kronecker delta
γ	circular frequency of axial oscillation of moving loads
$\omega_i\,(i = 1, 2)$	two circular frequencies of axisymmetric modes
$\Omega_i\,(i = 1, 2, 3)$	three circular frequencies of non-axisymmetric modes
$\lambda = m\pi a/l$	dimensionless axial wavelength parameter
ρ	mass density of shell
ν	Poisson's ratio

1. INTRODUCTION

THE problem of the dynamic response of cylindrical shells has received considerable attention in recent years. Particularly, the response of thin circular cylindrical shells to travelling shock waves has been the subject of many investigations. However, most of the studies on moving loads, such as those given in Refs. [1–5], were concerned with axially symmetric, uniformly moving loads on cylindrical shells which are free of initial stress. Reismann [6] and Herrmann and Baker [7] presented solutions for the response of pressurized cylindrical shells subjected to uniformly travelling axially symmetric loads. Schlack and Raske [8] obtained resonance conditions and response characteristics for a beam subjected to a cyclic moving load. Crocker [9, 10] presented a solution for the response of a simply supported flat panel subjected to oscillating shock waves of constant magnitude, and recently, Kessel and Schlack [11], by neglecting the longitudinal inertia, obtained an approximate solution for the response of a simply supported thin shell with uniform initial axial stress subjected to an oscillating ring load of constant intensity. However, no known solutions derived from shell theory including radial and in-plane inertia forces have been presented for the cases of an oscillating ring load as well as an axially cyclic moving radial point force for either unpressurized or pressurized shells. Loadings of this nature have become increasingly important in several fields of engineering, among others, in the reality of skin structure of an aerospace vehicle, jet engine intakes of supersonic aircraft subjected to oscillating shock pressure fronts and to cyclic travelling localized pressure discontinuities and surface finishing operations.

Although Cottis [12, 13], based on Reissner's shallow shell assumption, succeeded in obtaining the integral representation for only the radial component of displacement for both isotropic and orthotropic finite, simply supported cylindrical shells due to radial loadings by utilizing the Green's function technique, no known expressions of Green's functions for vector-valued displacement functions, i.e. Green's tensor based upon thin shell theories retaining all translatory inertia forces in the radial, axial and circumferential directions have been explored in the literature.

It is the main purpose of this paper to derive the integral representation for the displacement vector of the mid-surface of a prestressed thin circular cylindrical shell subjected to arbitrary three dimensional time-dependent surface loads and to present the associated Green's tensor kernel. The shells considered herein are of finite length with simply supported ends. The integral solution is general in the sense that it is applicable to shells under arbitrary loadings and reduces the problem of solving the partial differential equations to a much more straightforward integration.

Further, this paper presents the analytical solutions to the specific problems of an oscillating ring load and an axially cyclic moving radial point force on pressurized shells as direct applications of the general integral solution.

It is pointed out that although the analysis is based on Flugge's equations, the derivation and solutions are valid for Donnell's equations as well, provided that the definitions of some coefficients are properly adjusted as shown in the Appendix.

2. BASIC EQUATIONS

In Fig. 1, l, a, h are the length, mean radius and thickness respectively of the cylinder; x, y, z are the cylindrical coordinates for the mid-surface of the cylinder; u, v, w are the corresponding displacement components of a point on the mid-surface; α, β are dimensionless coordinates denoted by $\alpha = x/a$, $\beta = y/a$ and q_1, q_2, q_3 are the arbitrary time dependent surface loadings along x, y, z directions, respectively.

The constants N_1^0, N_2^0 represent the uniform initial membrane stress resultants per unit length in the cylinder along the x, y directions respectively due to initial internal and/or external pressure and a system of static loadings.

In addition to the conventional thin shell assumptions of homogeneity, isotropy, small displacements and the thickness–radius ratio less than 1/10, this paper assumes the transverse shear deformation and rotatory inertia are negligible, and the initial stress resultants N_1^0 and N_2^0 are less than the static buckling loads. Damping is also excluded from this investigation.

According to Flugge [14], the governing equations of motion for the prestressed circular cylindrical shell may be written as

$$R\vec{d}(x, \beta, t) = -\frac{a^2}{F}\vec{q}(x, \beta, t) \tag{1}$$

where R, $\vec{d}$ and $\vec{q}$ are matrices of partial differential operators, displacement and forcing vectors, respectively, as given by

$$R = \begin{bmatrix} A_{11}-\dfrac{\rho h a^2}{F}\dfrac{\partial^2}{\partial t^2} & A_{12} & A_{13} \\ A_{21} & A_{22}-\dfrac{\rho h a^2}{F}\dfrac{\partial^2}{\partial t^2} & A_{23} \\ -A_{31} & -A_{32} & -A_{33}-\dfrac{\rho h a^2}{F}\dfrac{\partial^2}{\partial t^2} \end{bmatrix}. \tag{2}$$

$$\vec{d} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}; \quad \vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix},$$

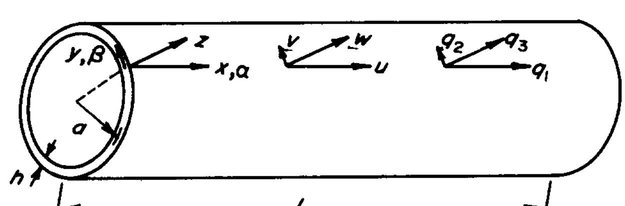

Fig. 1. Sign convention for coordinates, displacements and loads.

in which ρ is mass density of shell, t is time and A_{ij} are differential operators given by

$$
\begin{aligned}
A_{11} &= (1+\alpha_1)\frac{\partial^2}{\partial\alpha^2}+\left\{\left(\frac{1-\nu}{2}\right)(1+k)+\alpha_z\right\}\frac{\partial^2}{\partial\beta^2} \\
A_{12} &= \left(\frac{1+\nu}{2}\right)\frac{\partial^2}{\partial\alpha\,\partial\beta} = A_{21} \\
A_{13} &= (\nu-\alpha_2)\frac{\partial}{\partial\alpha}-k\frac{\partial^3}{\partial\alpha^3}+k\left(\frac{1-\nu}{2}\right)\frac{\partial^3}{\partial\alpha\,\partial\beta^2} = A_{31} \\
A_{22} &= (1+\alpha_2)\frac{\partial^2}{\partial\beta^2}+\left\{\frac{1-\nu}{2}+k\tfrac{3}{2}(1-\nu)+\alpha_1\right\}\frac{\partial^2}{\partial\alpha^2} \\
A_{23} &= (1+\alpha_2)\frac{\partial}{\partial\beta}-k\left(\frac{3-\nu}{2}\right)\frac{\partial^3}{\partial\alpha^2\alpha\beta} = A_{32} \\
A_{33} &= 1+k+k\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)^2-\alpha_1\left(\frac{\partial^2}{\partial\alpha^2}\right)+(2k-\alpha_2)\frac{\partial^2}{\partial\beta^2},
\end{aligned}
\tag{3}
$$

where

$$
k = \frac{h^2}{12a^2}, \qquad F = \frac{Eh}{1-\nu^2}, \qquad \alpha_1 = (N_1^0)/F, \qquad \alpha_2 = (N_2^0)/F.
$$

The shell with simply supported ends has the homogeneous boundary conditions

$$
v = w = \frac{\partial u}{\partial x} = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0 \text{ and } l, \tag{4}
$$

and for simplicity, homogeneous initial displacements and velocities in all three directions are assumed

$$
\vec{d}(x,\beta,0) = \frac{\partial}{\partial t}\vec{d}(x,\beta,0) = 0. \tag{5}
$$

3. GENERAL SOLUTION—GREEN'S FUNCTION TECHNIQUE

Equation (1) together with the boundary and initial conditions equations (4) and (5) defines the dynamic problem for an arbitrary forcing field $\vec{q}(x,\beta,t)$.

Associated with this problem there is a Green's tensor

$$
G(x,\beta,t;x_0,\beta_0,t_0) = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}, \tag{6}
$$

in which $K_{ij}(x,\beta,t;x_0,\beta_0,t_0)$ is physically interpreted as the response component in the ith direction of a point x, β on the shell mid-surface at a time t due to an impulse at a point x_0, β_0 in the jth direction at a prior time t_0. Since we are interested in a solution of equation (1) for only $t > 0$, it should then be kept in mind that $t_0 > 0$ is imposed hereafter.

Inspection of equation (1) suggests that the Green's tensor K_{ij} satisfies the following equations:

$$R\begin{bmatrix} K_{11} \\ K_{21} \\ K_{31} \end{bmatrix} = -\frac{a}{F}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p. \tag{7}$$

$$R\begin{bmatrix} K_{12} \\ K_{22} \\ K_{32} \end{bmatrix} = -\frac{a}{F}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} p. \tag{8}$$

$$R\begin{bmatrix} K_{13} \\ K_{23} \\ K_{33} \end{bmatrix} = -\frac{a}{F}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} p, \tag{9}$$

where

$$p = \delta(x-x_0)\delta(\beta-\beta_0)\delta(t-t_0) \tag{10}$$

and δ is the Dirac distribution in which x_0, β_0, t_0 is the impulse source point with $0 < x_0 < l$ and $t_0 > 0$. The Green's tensor also satisfies the same homogeneous boundary conditions as those for $\vec{d}$

$$K_{2i} = K_{3i} = \frac{\partial K_{1i}}{\partial x} = \frac{\partial^2 K_{3i}}{\partial x^2} = 0 \quad \text{at } x = 0 \text{ and } l \tag{11}$$

$$\text{for } i = 1, 2, 3;$$

and the causality conditions

$$K_{ij}(x, \beta, t; x_0, \beta_0, t_0) = \frac{\partial}{\partial t}K_{ij}(x, \beta, t; x_0, \beta_0, t_0) = 0 \quad \text{if } t < t_0, \quad \text{for } i, j = 1, 2, 3. \tag{12}$$

It is plausible at this moment to think that certain relations of reciprocity property for Green's tensor exist on account of symmetry in operators A_{ij} in the sense of Hilbert space which has the homogeneous boundary conditions equation (11). The rest of this section will be devoted to the determination of the Green's tensor and its reciprocity relation as well as the derivation of general solution to the dynamic problem defined by equations (1), (4) and (5) in terms of Green's tensor.

A. Determination of Green's tensor

In view of the boundary conditions equation (11), the solution of equation (9) may be sought in the form

$$\begin{aligned} K_{13}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} U_3 \cos(m\pi x/l)\cos n(\beta-\beta_0) \\ K_{23}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_3 \sin(m\pi x/l)\sin n(\beta-\beta_0) \\ K_{33}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} W_3 \sin(m\pi x/l)\cos n(\beta-\beta_0), \end{aligned} \tag{13}$$

where U_3, V_3, W_3 are functions of t, t_0, x_0 and dependent on integers m, n. They are to be so chosen as to satisfy the equation of motion equation (9) and the initial conditions implied by equation (12).

Substituting equation (13) into equation (9) and multiplying the first, second and third equations by $[\cos(m\pi x/l)\cos n(\beta-\beta_0)]$, $[\sin(m\pi x/l)\sin n(\beta-\beta_0)]$ and $[\sin(m\pi x/l)\cos n(\beta-\beta_0)]$ respectively, then integrating the resulting equations over the shell surface, one obtains after rearranging terms the following two categories:

(a) Axisymmetrical modes

$$\frac{\rho h a^2}{F}\begin{bmatrix}\ddot{U}_3\\ \ddot{W}_3\end{bmatrix}+\begin{bmatrix}A_4 & -A_5\\ -A_5 & C_4\end{bmatrix}\begin{bmatrix}U_3\\ W_3\end{bmatrix}=\frac{a}{Fl\pi}\sin\frac{m\pi x_0}{l}\begin{bmatrix}0\\ \delta(t-t_0)\end{bmatrix}, \tag{14}$$

for $n = 0$, $m = 1, 2, \ldots$.

(b) Non-axisymmetric modes

$$\frac{\rho h a^2}{F}\begin{bmatrix}\ddot{U}_3\\ \ddot{V}_3\\ \ddot{W}_3\end{bmatrix}+\begin{bmatrix}A_1 & -A_2 & -A_3\\ -A_2 & B_2 & B_3\\ -A_3 & B_3 & C_3\end{bmatrix}\begin{bmatrix}U_3\\ V_3\\ W_3\end{bmatrix}=\frac{2a}{Fl\pi}\sin\frac{m\pi x_0}{l}\begin{bmatrix}0\\ 0\\ \delta(t-t_0)\end{bmatrix}, \tag{15}$$

for $m, n = 1, 2, \ldots$.

where $(\ddot{\ })$ is used to denote the second derivative with respect to time t, and the coefficients A_i, B_i, C_i are defined by

$$\begin{aligned}
A_0 &= \left(\frac{1-\nu}{2}\right)n^2+\left(\frac{1-\nu}{2}\right)n^2k+n^2\alpha_2\\
A_1 &= (1+\alpha_1)\lambda^2+\left\{\left(\frac{1-\nu}{2}\right)(1+k)+\alpha_2\right\}n^2\\
A_2 &= \left(\frac{1+\nu}{2}\right)\lambda n, \qquad A_3 = \left[(\nu-\alpha_2)\lambda+k\lambda^3-k\left(\frac{1-\nu}{2}\right)\lambda n^2\right]\\
A_4 &= [(1+\alpha_1)\lambda^2], \qquad A_5 = [(\nu-\alpha_2)\lambda+k\lambda^3]\\
B_0 &= \left[\left(\frac{1-\nu}{2}\right)+k\tfrac{3}{2}(1-\nu)+\alpha_1\right]\lambda^2\\
B_2 &= \left[(1+\alpha_2)n^2+\lambda^2\left\{\left(\frac{1-\nu}{2}\right)+k\tfrac{3}{2}(1-\nu)+\alpha_1\right\}\right]\\
B_3 &= \left[(1+\alpha_2)n+k\left(\frac{3-\nu}{2}\right)\lambda^2 n\right]\\
C_3 &= [1+k+k(\lambda^2+n^2)^2+\alpha_1\lambda^2+(\alpha_2-2k)n^2]\\
C_4 &= [1+k+k\lambda^4+\alpha_1\lambda^2],
\end{aligned} \tag{16}$$

where $\lambda = m\pi a/l$.

Employing Laplace transforms with respect to t to equations (14) and (15) then taking the inversion of the resulting transformed equations in combination with the convolution theorem, one finds the expressions for the time functions as

$$\begin{bmatrix} U_3 \\ W_3 \end{bmatrix} = \sum_{i=1}^{2} \frac{H(t-t_0)\sin\omega_i(t-t_0)\sin(m\pi x_0/l)}{\rho hal\pi(\omega_j^2-\omega_i^2)\omega_i} \begin{bmatrix} a_5 \\ a_4-\omega_i^2 \end{bmatrix} \tag{17}$$

for

$$n = 0, \qquad m = 1, 2, \ldots, \qquad i \neq j;$$

and

$$\begin{bmatrix} U_3 \\ V_3 \\ W_3 \end{bmatrix} = \sum_{i=1}^{3} \frac{2H(t-t_0)\sin\Omega_i(t-t_0)\sin(m\pi x_0/l)}{\rho hal\pi(\Omega_i^2-\Omega_j^2)(\Omega_i^2-\Omega_k^2)\Omega_i} \begin{bmatrix} g_1-a_3\Omega_i^2 \\ g_2+b_3\Omega_i^2 \\ \Omega_i^4-f_5\Omega_i^2+f_6 \end{bmatrix} \tag{18}$$

for

$$m, n = 1, 2, \ldots, \qquad i \neq j \neq k \neq i;$$

where $H(t-t_0)$ is the Heaviside step function with a discontinuity at t_0, and the coefficients are given by

$$\begin{aligned} &a_i = \{F/(\rho ha^2)\}A_i; \qquad b_i = \{F/(\rho ha^2)\}B_i; \qquad c_i = \{F/(\rho ha^2)\}C_i \\ &f_1 = b_2+c_3, \qquad f_2 = b_2c_3-b_3^2, \qquad f_3 = a_1+c_3 \\ &f_4 = a_1c_3-a_3^2, \qquad f_5 = a_1+b_2, \qquad f_6 = a_1b_2-a_2^2 \\ &g_0 = a_2c_3-a_3b_3, \qquad g_1 = a_3b_2-a_2b_3, \qquad g_2 = a_2a_3-a_1b_3, \end{aligned} \tag{19}$$

ω_i^2 $(i = 1, 2)$ are the two roots of the frequency equation for axisymmetric modes of free vibration

$$(\omega^2)^2-(a_4+c_4)\omega^2+(a_4c_4-a_5^2) = 0 \tag{20}$$

and Ω_1^2, Ω_2^2 and Ω_3^2 are the three roots of the frequency equation for non-axisymmetric modes of free vibration of the cylinder

$$(\omega^2)^3-e_2(\omega^2)^2+e_1\omega^2-e_0 = 0 \tag{21}$$

where

$$e_0 = \begin{bmatrix} a_1 & -a_2 & -a_3 \\ -a_2 & b_2 & b_3 \\ -a_3 & b_3 & c_3 \end{bmatrix} \tag{22}$$

$$e_1 = a_1b_2+a_1c_3+b_2c_3-a_3^2-a_2^2-b_3^2; \qquad e_2 = a_1+b_2+c_3.$$

It should be noted that the use of the following initial conditions derived from equation (12)

$$\begin{aligned} &U_3 = V_3 = W_3 = 0 \\ &\dot{U}_3 = \dot{V}_3 = \dot{W}_3 = 0 \quad \text{at } t = 0 \end{aligned} \tag{23}$$

and the frequency inequalities*

$$\omega_2^2 > \omega_1^2 > 0 \qquad \text{for } n = 0, m = 1, 2 \ldots$$
$$\Omega_3^2 > \Omega_2^2 > \Omega_1^2 > 0 \quad \text{for } m, n = 1, 2 \ldots$$

has been made in arriving at equations (17) and (18).

Equation (13) together with equations (17) and (18) determines the Green's tensor components K_{i3} $(i = 1, 2, 3)$.

In a similar way, K_{i1} $(i = 1, 2, 3)$, the solution for equation (7) may be expressed as

$$\begin{aligned} K_{11}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} U_1 \cos(m\pi x/l) \cos n(\beta - \beta_0) \\ K_{21}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_1 \sin(m\pi x/l) \sin n(\beta - \beta_0) \\ K_{31}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} W_1 \sin(m\pi x/l) \cos n(\beta - \beta_0) \end{aligned} \tag{24}$$

where the time coefficients are given by

$$\begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = \sum_{i=1}^{3} \frac{2H(t-t_0)\sin \Omega_i(t-t_0)\cos(m\pi x_0/l)}{\rho h a l \pi(\Omega_i^2 - \Omega_j^2)(\Omega_i^2 - \Omega_k^2)\Omega_i} \begin{bmatrix} \Omega_i^4 - f_1\Omega_i^2 + f_2 \\ g_0 - a_2\Omega_i^2 \\ g_1 - a_3\Omega_i^2 \end{bmatrix} \tag{25}$$

$$i \neq j \neq k \neq i; \quad \text{for} \quad m, n = 1, 2 \ldots$$

$$\begin{bmatrix} U_1 \\ W_1 \end{bmatrix} = \sum_{i=1}^{2} \frac{H(t-t_0)\sin \omega_i(t-t_0)\cos(m\pi x_0/l)}{\rho h a l \pi(\omega_j^2 - \omega_i^2)\omega_i} \begin{bmatrix} c_4 - \omega_i^2 \\ a_5 \end{bmatrix} \tag{26}$$

$$i \neq j; \quad \text{for } n = 0, m = 1, 2 \ldots,$$

and

$$U_1 = \frac{H(t-t_0)\sin \omega_{0n}(t-t_0)}{\rho h a l \pi \omega_{0n}} \quad \text{for } m = 0, \quad n = 1, 2, \ldots \tag{27}$$

in which $\omega_{0n}^2 = a_0$ is frequency squared of longitudinal modes at which all particles along $\beta = \beta_1$ vibrate with equal amplitude and phase longitudinally for each n. The rigid body mode U_1 $(m = n = 0)$ which has no relevance to the vibratory motion has been excluded from this analysis.

Similarly, K_{i2} $(i = 1, 2, 3)$ are given by

$$\begin{aligned} K_{12}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_2 \cos(m\pi x/l) \sin n(\beta - \beta_0) \\ K_{22}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} V_2 \sin(m\pi x/l) \cos n(\beta - \beta_0) \\ K_{32}(x, \beta, t; x_0, \beta_0, t_0) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_2 \sin(m\pi x/l) \sin n(\beta - \beta_0), \end{aligned} \tag{28}$$

* For detailed proof refer to Liao [15].

where the time coefficients are given as

$$\begin{bmatrix} U_2 \\ V_2 \\ W_2 \end{bmatrix} = \sum_{i=1}^{3} \frac{2H(t-t_0)\sin\Omega_i(t-t_0)\sin(m\pi x_0/l)}{\rho hal\pi(\Omega_i^2-\Omega_j^2)(\Omega_i^2-\Omega_k^2)\Omega_i} \begin{bmatrix} -(g_0-a_2\Omega_i^2) \\ \Omega_i^4-f_3\Omega_i^2+f_4 \\ -(g_2+b_3\Omega_i^2) \end{bmatrix} \tag{29}$$

$$i \neq j \neq k \neq i, \quad \text{for } m, n = 1, 2, \ldots$$

$$V_2 = \frac{H(t-t_0)\sin\omega_{m0}(t-t_0)\sin(m\pi x_0/l)}{\rho hal\pi\omega_{m0}} \quad \text{for } n = 0, \quad m = 1, 2, \ldots \tag{30}$$

where $\omega_{m0}^2 = b_0$ is the frequency squared of the torsional modes at which all particles at any cross section which is perpendicular to the axis of symmetry of the cylinder vibrate torsionally with equal phase and amplitude for each m.

A study of equations (13), (17), (18) and equations (24) through (30) reveals the reciprocity relations of the Green's tensor as

$$K_{ij}(x, \beta, t; x_0, \beta_0, t_0) = K_{ji}(x_0, \beta_0, -t_0; x, \beta, -t) \qquad i, j = 1, 2, 3; \tag{31}$$

or in the form

$$K_{ij}(x, \beta; x_0, \beta_0; t-t_0) = K_{ji}(x_0, \beta_0; x, \beta; t-t_0) \qquad i, j = 1, 2, 3. \tag{32}$$

B. Derivation of general solution

For convenience, equations (7), (8) and (9) are combined into a single matrix equation as

$$RG(x, \beta, t; x_0, \beta_0, t_0) = -\frac{a}{F}E'p(x, \beta, t; x_0, \beta_0, t_0), \tag{33}$$

where R, G, p are defined in equations (2), (6), (10) respectively, and E' is the identity matrix of order 3. Equation (33) may be put in the component form

$$R_{ij}K_{jk}(x, \beta, t; x_0, \beta_0, t_0) = -\frac{a}{F}\delta_{ik}p(x, \beta, t; x_0, \beta_0, t_0) \tag{34}$$

where δ_{ik} is the Kronecker delta symbol and

$$\begin{aligned} R_{ij} &= A_{ij} - \delta_{ij}(\rho ha^2/F)\frac{\partial^2}{\partial t^2} \qquad \text{for } i = 1, 2 \\ &= -A_{ij} - \delta_{ij}(\rho ha^2/F)\frac{\partial^2}{\partial t^2} \quad \text{for } i = 3. \end{aligned} \tag{35}$$

The integral representation of the solution for equation (1) may now be derived in a similar procedure as described by Morse and Feshbach [16] for uncoupled wave equation with further generalization fitted for a system of coupled differential equations as follows: Write equations (1) and (33) in the x_0, β_0, t_0 coordinates, then premultiply the first of these equations by Green's tensor $G(x, \beta, t; x_0, \beta_0, t_0)$ and the second by a row matrix $[u \;\; v \;\; w]$ in the x_0, β_0, t_0 coordinates, respectively; transpose the latter matrix equation, making use of the reciprocity relation equation (31), and integrate the resulting equations over the surface under investigation and over time from 0 to t^+. The result is

$$-\frac{a^2}{F}\int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 G(x,\beta,t;x_0,\beta_0,t_0)\vec{q}(x_0,\beta_0,t_0) = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{36}$$

$$-\frac{a^2}{F}\vec{d}(x,\beta,t) = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} \tag{37}$$

where

$$M_i = u_k R_{kj} K_{ji}(x_0,\beta_0,-t_0;x,\beta,-t); \qquad i = 1,2,3, \tag{38}$$

$$N_i = K_{ij}(x,\beta,t;x_0,\beta_0,t_0) R_{jk} u_k; \qquad i = 1,2,3, \tag{39}$$

where the summation notation for repeated indices has been utilized and u_k, R_{kj}, R_{jk} are all in the x_0, β_0, t_0 coordinates. In equations (36) and (37) the surface element $\mathrm{d}S_0$ is used to denote $(a\,\mathrm{d}x_0\,\mathrm{d}\beta_0)$, and the symbol t^+ has been utilized to represent $t+\varepsilon$ where ε is an arbitrary small positive number so as to avoid ending the integration exactly at the singularity of the Dirac distribution.

Subtracting equation (37) from equation (36) yields the complete solution in the integral form

$$\vec{d}(x,\beta,t) = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 G(x,\beta,t;x_0,\beta_0,t_0)\vec{q}(x_0,\beta_0,t_0) + \frac{F}{a^2}\int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 \begin{bmatrix} N_1 - M_1 \\ N_2 - M_2 \\ N_3 - M_3 \end{bmatrix}. \tag{40}$$

By substituting equations (38) and (39) into equation (40) and noting equations (31) and (35), equation (40) may be conveniently put in the component form

$$\begin{aligned} u_i(x,\beta,t) = &\int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 K_{ij}(x,\beta,t;x_0,\beta_0,t_0) q_j(x_0,\beta_0,t_0) \\ &+ \frac{F}{a^2}\int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 \{K_{ij}(\pm A_{jk})u_k - u_k(\pm A_{kj})K_{ij}\} \\ &- \rho h \int \mathrm{d}S_0 \int_0^{t^+} \mathrm{d}t_0 \left\{K_{ij}\delta_{jk}\frac{\partial^2}{\partial t_0^2}u_k - u_k\delta_{kj}\frac{\partial^2}{\partial t_0^2}K_{ij}\right\}, \end{aligned} \tag{41}$$

for $i = 1,2,3$;

where K_{ij} are functions of $(x,\beta,t;x_0,\beta_0,t_0)$, and for brevity the notation is introduced

$$\begin{aligned} (\pm A_{kj}) &= A_{kj} && \text{if } k = 1,2 \\ &= (-A_{kj}) && \text{if } k = 3. \end{aligned} \tag{42}$$

Equation (40) or equation (41) gives the complete solution to the inhomogeneous problem described by equation (1) with inclusion of the satisfaction of inhomogeneous boundary conditions and nontrivial initial conditions of displacements u_i. The first term of equation (41) represents the effect of impulse sources; the second the effect of the boundary conditions, while the last term includes the effect of initial conditions of u_i.

If our attention is restricted to the problem defined by equation (1) along with the homogeneous boundary and initial conditions equations (4) and (5), it is seen by successive partial integration with respect to time variable the third integral of equation (41) vanishes since

$$\begin{aligned} &\int \mathrm{d}S_0 \int_0^{t^+} \mathrm{d}t_0 K_{ij}\delta_{jk} \frac{\partial^2}{\partial t_0^2} u_k \\ &= \int \mathrm{d}S_0 \delta_{jk} \left\{ K_{ij} \frac{\partial u_k}{\partial t_0}\bigg|_0^{t^+} - u_k \frac{\partial K_{ij}}{\partial t_0}\bigg|_0^{t^+} \right\} + \int \mathrm{d}S_0 \int_0^{t^+} \mathrm{d}t_0 u_k \delta_{kj} \frac{\partial^2}{\partial t_0^2} K_{ij} \\ &= \int \mathrm{d}S_0 \int_0^{t^+} \mathrm{d}t_0 u_k \delta_{kj} \frac{\partial^2}{\partial t_0^2} K_{ij}, \end{aligned} \tag{43}$$

in which the initial conditions equation (5) and the causality conditions equation (12) are utilized.

Also, by successive partial integration with respect to space variables x_0 and/or β_0 one may conclude

$$\int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 K_{ij}(\pm A_{jk}) u_k = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 u_k(\pm A_{kj}) K_{ij} \tag{44}$$

$$\text{for } i = 1, 2, 3, \text{ with no sum on } k, j;$$

by virtue of the homogeneous boundary conditions satisfied by both $\vec{d}$ and K_{ij}. As a consequence, the second integral of equation (41) vanishes also. It thus yields the integral representation of the general solution as

$$\vec{d}(x, \beta, t) = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 G(x, \beta, t; x_0, \beta_0, t_0) \vec{q}(x_0, \beta_0, t_0) \tag{45}$$

or in the component form

$$u_i(x, \beta, t) = \int_0^{t^+} \mathrm{d}t_0 \int \mathrm{d}S_0 K_{ij}(x, \beta, t; x_0, \beta_0, t_0) q_j(x_0, \beta_0, t_0). \tag{46}$$

In the case of $q_1 = q_2 = 0$, the radial load q_3 is the only force acting on the shell, the solution becomes

$$u_i(x, \beta, t) = \int_0^{t^+} \int_0^{2\pi} \int_0^{l} K_{i3}(x, \beta, t; x_0, \beta_0, t_0) q_3(x_0, \beta_0, t_0) a \, \mathrm{d}x_0 \, \mathrm{d}\beta_0 \, \mathrm{d}t_0 \tag{47}$$

$$\text{for } i = 1, 2, 3.$$

4. ANALYTICAL SOLUTIONS FOR CASES OF CYCLIC MOVING LOADS

A. Oscillating moving ring load

An axially symmetric moving ring load oscillates about a fixed parallel of the cylinder $x = x_1$ as shown in Fig. 2. Then

$$
\begin{aligned}
q_1 &= q_2 = 0 \\
q_3 &= Q_0 \cos et \;\; \delta(x-\{x_1+X \sin \gamma t\}) \quad \text{for all } t \geqq 0,
\end{aligned} \tag{48}
$$

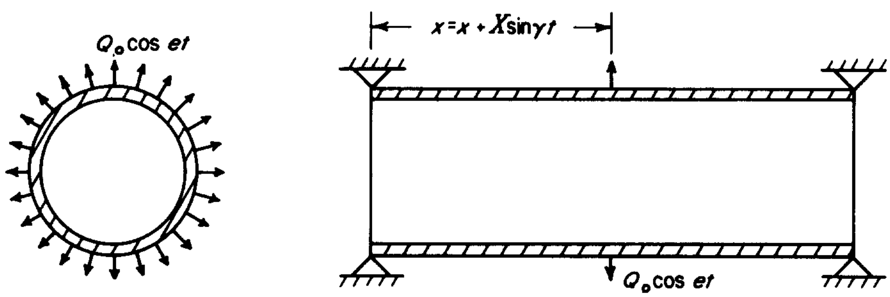

FIG. 2. Load geometry of oscillating ring load.

where e is the transverse frequency of ring load; γ is the circular frequency of the axial oscillation of load; X is the amplitude of axially oscillatory position of ring load and Q_0 is the amplitude of intensity of ring load, for brevity, the conditions

$$
0 \leqq (x_1 \pm X) \leqq l \quad \text{and} \quad x_1 > 0
$$

are assumed.

By noting equations (13), (17) and (18), the displacement equation (47) becomes

$$
\begin{aligned}
v &= 0 \\
u &= \sum_{m=1}^{\infty} \sum_{i=1}^{2} \frac{a_5 \cos(m\pi x/l)}{\rho h(\omega_j^2-\omega_i^2)\omega_i} I_i(t) \\
w &= \sum_{m=1}^{\infty} \sum_{i=1}^{2} \frac{(a_4-\omega_i^2) \sin(m\pi x/l)}{\rho h(\omega_j^2-\omega_i^2)\omega_i} I_i(t); \qquad i \neq k
\end{aligned} \tag{49}
$$

where

$$
I_i(t) = \frac{2Q_0}{l} \int_0^t \cos e\tau \sin \frac{m\pi}{l}(x_1+X \sin \gamma\tau) \sin \omega_i(t-\tau)\, \mathrm{d}\tau. \tag{50}
$$

Noting the following Bessel function identities [17]

$$
\begin{aligned}
\cos(G\sin H) &= J_0(G)+2\sum_{j=1}^{\infty}\{J_{2j}(G)\cos 2jH\}\\
\sin(G\sin H) &= 2\sum_{j=0}^{\infty}\{J_{2j+1}(G)\sin(2j+1)H\}\\
\cos(G\cos H) &= J_0(G)+2\sum_{j=1}^{\infty}\{(-1)^j J_{2j}(G)\cos 2jH\}\\
\sin(G\cos H) &= 2\sum_{j=0}^{\infty}\{(-1)^j J_{2j+1}(G)\cos(2j+1)H\},
\end{aligned}
\tag{51}
$$

where J is the Bessel function of the first kind, one may evaluate the integral of equation (50).

$$
\begin{aligned}
I_i(t) = {} & \frac{Q_0}{l}J_0(M)\sin L\left\{\frac{\cos et-\cos\omega_i t}{\omega_i-e}+\frac{\cos et-\cos\omega_i t}{\omega_i+e}\right\}\\
& +\frac{Q_0}{l}\sin L\sum_{j=1}^{\infty}J_{2j}(M)\left\{\frac{\cos(e+2j\gamma)t-\cos\omega_i t}{\omega_i-e-2j\gamma}+\frac{\cos(e-2j\gamma)t-\cos\omega_i t}{\omega_i+e-2j\gamma}\right.\\
& \left.+\frac{\cos(e-2j\gamma)t-\cos\omega_i t}{\omega_i-e+2j\gamma}+\frac{\cos(e+2j\gamma)t-\cos\omega_i t}{\omega_i+e+2j\gamma}\right\}\\
& +\frac{Q_0}{l}\cos L\sum_{j=0}^{\infty}J_{2j+1}(M)\left\{\frac{\sin[-e+(2j+1)\gamma]t+\sin\omega_i t}{\omega_i-e+(2j+1)\gamma}\right.\\
& +\frac{\sin[e+(2j+1)\gamma]t+\sin\omega_i t}{\omega_i+e+(2j+1)\gamma}+\frac{\sin[e+(2j+1)\gamma]t-\sin\omega_i t}{\omega_i-e-(2j+1)\gamma}\\
& \left.+\frac{\sin[-e+(2j+1)\gamma]t-\sin\omega_i t}{\omega_i+e-(2j+1)\gamma}\right\},
\end{aligned}
\tag{52}
$$

where

$$
L = m\pi x_1/l, \qquad M = m\pi X/l. \tag{53}
$$

An investigation of equations (52) and (49) shows that resonance will occur whenever any of the following conditions is satisfied for every value of m:

$$
\begin{aligned}
\omega_i &= e & &\\
|\omega_i-e| &= 2j\gamma; & j &= 1,2,3,\ldots\\
\omega_i+e &= 2j\gamma; & j &= 1,2,3,\ldots\\
\omega_i+e &= (2j+1)\gamma; & j &= 0,1,2,\ldots\\
|\omega_i-e| &= (2j+1)\gamma; & j &= 0,1,2,\ldots
\end{aligned}
\tag{54}
$$

where $e>0$, $\gamma>0$.

Equation (54) may be rewritten as

$$\begin{aligned} \omega_i &= e \\ |\omega_i \pm e| &= j\gamma; \qquad j = 1, 2, \ldots \\ i &= 1, 2 \quad \text{for } m = 1, 2, \ldots \end{aligned} \tag{55}$$

Considering equation (52), it is noticed that for $e = 0$ the resonance conditions become

$$(\omega_i)/j = \gamma, \qquad j = 1, 2, \ldots \qquad \text{for } i = 1, 2; \quad m = 1, 2, \ldots. \tag{56}$$

These results agree with those presented by Crocker [10] for a flat panel and Kessel and Schlack [11] for a beam subjected to a similar type of loading.

B. Radial point load oscillating axially about a fixed point

As shown schematically in Fig. 3, the loading function may be written as

$$\begin{aligned} q_1 &= q_2 = 0 \\ q_3 &= -P_0\delta(x - \{x_1 + X \sin \gamma t\})\delta(y - 0) \qquad t \geqq 0 \end{aligned} \tag{57}$$

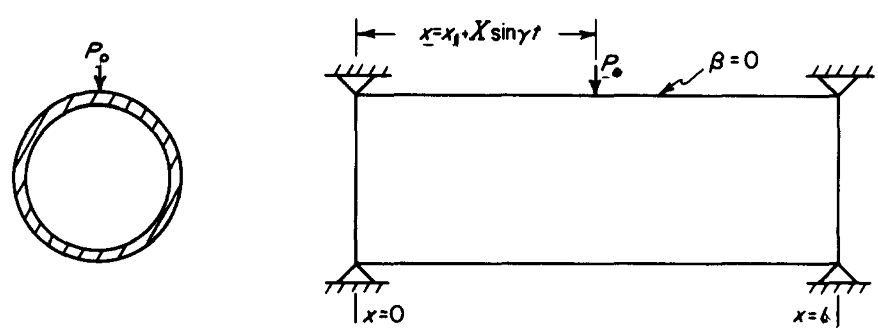

FIG. 3. Load geometry of cyclic moving point force.

where P_0 is the concentrated constant force whose position is described by $(x_1 + X \sin \gamma t, 0)$ in which x_1 is the axial coordinate of the center of oscillation, X is the amplitude of axial oscillating movement and γ the circular frequency of axial oscillation of the concentrated force.

For simplicity, the consideration is restricted to $x_1 > 0$, $0 \leqq x_1 + X \sin \gamma t \leqq l$.

Substituting equation (57) into equation (47), one obtains

$$
\begin{aligned}
u &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3} -(g_1 - a_3\Omega_i^2)2P_0 \cos\frac{m\pi x}{l}\cos n\beta T_i(\Omega)Q_i(t,\Omega) \\
&\quad + \sum_{m=1}^{\infty}\sum_{i=1}^{2} -a_5 P_0 \cos\frac{m\pi x}{l} S_i(\omega)Q_i(t,\omega) \\
v &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3} -(g_2 + b_3\Omega_i^2)2P_0 \sin\frac{m\pi x}{l}\sin n\beta T_i(\Omega)Q_i(t,\Omega) \\
w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3} -(\Omega_i^4 - f_5\Omega_i^2 + f_6)2P_0 \sin\frac{m\pi x}{l}\cos n\beta T_i(\Omega)Q_i(t,\Omega) \\
&\quad + \sum_{m=1}^{\infty}\sum_{i=1}^{2} -(a_4 - \omega_i^2)P_0 \sin\frac{m\pi x}{l} S_i(\omega)Q_i(t,\omega),
\end{aligned}
\tag{58}
$$

where

$$
\begin{aligned}
1/S_i(\omega) &= \rho hal\pi(\omega_j^2 - \omega_i^2)\omega_i \\
1/T_i(\Omega) &= \rho hal\pi(\Omega_i^2 - \Omega_j^2)(\Omega_i^2 - \Omega_k^2)\Omega_i \qquad i \neq j \neq k \neq i,
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
Q_i(t,\Omega) &= \int_0^t \sin\frac{m\pi}{l}(x_1 + X\sin\gamma\tau)\sin\Omega_i(t-\tau)\,\mathrm{d}\tau \\
Q_i(t,\omega) &= \int_0^t \sin\frac{m\pi}{l}(x_1 + X\sin\gamma\tau)\sin\omega_i(t-\tau)\,\mathrm{d}\tau.
\end{aligned}
\tag{60}
$$

By noting equations (50), (52) and (53), it may be readily seen that equation (60) becomes

$$
\begin{aligned}
Q_i(t,\Omega) &= \int_0^t \sin(M + L\sin\gamma\tau)\sin\Omega_i(t-\tau)\,\mathrm{d}\tau = J_0(M)\sin L(1-\cos\Omega_i t)/\Omega_i \\
&\quad + \sin L\sum_{j=1}^{\infty} J_{2j}(M)(\cos 2j\gamma t - \cos\Omega_i t)2\Omega_i/[\Omega_i^2 - (2j\gamma)^2] \\
&\quad + \cos L\sum_{j=0}^{\infty} J_{2j+1}(M)\left[\frac{\sin(2j+1)\gamma t + \sin\Omega_i t}{\Omega_i + (2j+1)\gamma} + \frac{\sin(2j+1)\gamma t - \sin\Omega_i t}{\Omega_i - (2j+1)\gamma}\right].
\end{aligned}
\tag{61}
$$

Equations (58), (59) and (61) give the complete solution for this problem.

Equation (61) shows that this system has, theoretically, infinite opportunities of resonance for every mode of vibration:

$$
\begin{aligned}
|\gamma| &= \Omega_i/j, \qquad j = 1,2,3,\ldots \\
\text{for } i &= 1,2,3; \qquad m,n = 1,2,3,\ldots
\end{aligned}
\tag{62}
$$

and

$$
\begin{aligned}
|\gamma| &= \omega_i/j, \qquad j = 1,2,3,\ldots \\
\text{for } i &= 1,2; \qquad n = 0, \qquad m = 1,2,3,\ldots.
\end{aligned}
\tag{63}
$$

It is noticed that the terms containing $\sin\Omega_i t$, $\cos\Omega_i t$, $\sin\omega_i t$ or $\cos\omega_i t$ in equations (52) and (61) represent the effect of the free vibration on dynamic response, therefore the solutions provided by equation (49) together with equation (52) and by equation (58) with equations (59) and (61) to both problems are for transient response rather than steady state response. By disregarding these free vibration terms which in practical engineering systems are soon damped out, equations (52) and (61) become respectively

$$
\begin{aligned}
I_i(t) = \frac{2Q_0\omega_i}{l}\Bigg[& J_0(M)\sin L\cos et/(\omega_i^2-e^2) \\
& +\sin L\sum_{j=1}^{\infty} J_{2j}(M)\left\{\frac{\cos(e+2j\gamma)t}{\omega_i^2-(e+2j\gamma)^2}+\frac{\cos(e-2j\gamma)t}{\omega_i^2-(e-2j\gamma)^2}\right\} \\
& +\cos L\sum_{j=0}^{\infty} J_{2j+1}(M)\left\{\frac{\sin[-e+(2j+1)\gamma]t}{\omega_i^2-[e-(2j+1)\gamma]^2}+\frac{\sin[e+(2j+1)\gamma]t}{\omega_i^2-[e+(2j+1)\gamma]^2}\right\}\Bigg]
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
Q_i(t,\Omega) = {} & \frac{\sin L}{\Omega_i}J_0(M)+2\Omega_i\sin L\sum_{j=1}^{\infty} J_{2j}(M)\frac{\cos 2j\gamma t}{\Omega_i^2-(2j\gamma)^2} \\
& +2\Omega_i\cos L\sum_{j=0}^{\infty} J_{2j+1}(M)\frac{\sin(2j+1)\gamma t}{\Omega_i^2-(2j+1)^2\gamma^2}.
\end{aligned}
\tag{65}
$$

Therefore equations (64) and (49) give the steady state response for axially oscillating ring load problem, while equation (58) together with equations (59), (65) provides the steady state solution to the cyclic moving point force problem. The conditions for resonance equation (56) and equations (62), (63) remain unchanged.

5. NUMERICAL RESULTS

The lowest frequency of a shell will always occur when $m = 1$, and n either 2, 3, 4, ... depending on the shell's geometry.

Designating Ω_L as this lowest frequency, then from the results of Kessel and Schlack [11] in their study of a beam subject to a cyclic moving point force, we know that the most severe resonance will be excited by a load movement frequency $\gamma = \Omega_L$ or $\gamma = \Omega_L/2$ depending on the initial position axial coordinate x_1 and the amplitude of the axial movement X. For the special case of a concentrated load oscillating about the midspan of the shell along $\beta = 0$, and considering only the radial deflection of shell at point $x = l/2$, $\beta = 0$, then $x_1 = l/2$, then it follows that

$$
\begin{cases}
\sin\dfrac{m\pi x}{l} = \sin\dfrac{m\pi x_1}{l} = (-1)^{(m-1)/2} & \text{for } m = 1, 3, 5, \ldots \\[2ex]
\cos\dfrac{m\pi x_1}{l} = 0 & \text{for } m = 1, 3, 5, \ldots.
\end{cases}
$$

Therefore for $x = x_1 = l/2$, $\beta = 0$, utilizing the following dimensionless notations

$$\left.\begin{aligned} \tau &= \left[\frac{E}{\rho a^2(1-\nu^2)}\right]^{1/2} t; \qquad d_\gamma = \gamma^2\left[\frac{\rho a^2(1-\nu^2)}{E}\right] \\ d_i &= \Omega_i^2\left[\frac{\rho a^2(1-\nu^2)}{E}\right]; \qquad d_L = \Omega_L^2\left[\frac{\rho a^2(1-\nu^2)}{E}\right] \\ \bar{w} &= w\psi \end{aligned}\right\} \tag{66}$$

where

$$\psi = \frac{Elh\pi}{[P_0 a(1-\nu^2)]}$$

and after considerable mathematical manipulation, we have the steady state solution at resonance $\gamma = \Omega_L/2$, i.e. $d_\gamma = d_L/4$ for the particular mode $m = 1, i = 1$, for some n in the nondimensional form:

$$\bar{w} = \left\{\begin{array}{ll} \gamma \neq \dfrac{\Omega_i}{2} & \left\{\begin{aligned} &\sum_{m=1,3}^{\infty}\sum_{i=1}^{2}\frac{-(A_4-d_i)}{(d_j-d_i)d_i}\left[J_0(M)+\sum_{j=1}^{\infty}J_{2j}(M)\frac{2d_i\cos 2j\sqrt{d_\gamma}\tau}{(d_i-4j^2d_\gamma)}\right] \\ &+\sum_{m=1,3}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{3}\frac{-2(d_i^2-F_5d_i+F_6)}{(d_i-d_j)(d_i-d_k)d_i}\left[J_0(M)+\sum_{j=1}^{\infty}J_{2j}(M)\frac{2d_i\cos 2j\sqrt{d_\gamma}\tau}{(d_i-4j^2d_\gamma)}\right]\end{aligned}\right. \\ \gamma = \dfrac{\Omega_L}{2} & \left\{\begin{aligned} &+\frac{-2(d_L^2-F_5d_L+F_6)}{(d_L^2-d_2)(d_L-d_3)d_L}\Big[J_0(M_r)+J_2(M_r)(\sqrt{d_L}\tau\sin\sqrt{d_L}\tau) \\ &\qquad -J_4(M_r)\frac{2\cos 2\sqrt{d_L}\tau}{3}+\ldots\Big] \\ &+\sum_{i=2}^{3}\frac{-2(d_i^2-F_5d_i+F_6)}{(d_i-d_j)(d_i-d_k)d_i}\left[J_0(M_r)+\sum_{j=1}^{\infty}J_{2j}(M_r)\frac{2d_i\cos 2j\sqrt{d_\gamma}\tau}{(d_i-4j^2d_\gamma)}\right]\end{aligned}\right.\end{array}\right\} \tag{67}$$

where $M = m\pi X/l$ and $M_r = \pi X/l$. The static deflection due to a point force P_0 at $x_1 = l/2$ and $\beta = 0$ may be obtained from letting X approach zero, since

$$\bar{w}_s = \lim_{X\to 0} \bar{w}.$$

This result agrees with those given by Liao and Kessel [18].

The decaying nature of the Bessel functions J_{2n} in the numerator coupled with the rapid increase of shell frequency d_i in the denominator of equation (67) makes the series rapidly convergent. Numerical results from equation (67) are presented in Figs. 4–8.

Figures 4, 5 and 6 show the ratio of dynamic deflection to static deflection vs a dimensionless amplitude of load movement for different numbers of periods of dimensionless time. The shell parameters for Figs. 4, 5 and 6 are $h/a = 0{\cdot}01$, $L/a = 4{\cdot}0$ and respectively Fig. 4 with no initial stress, Fig. 5 with two-way initial tension and Fig. 6 with two-way initial compression. It is readily observed that large dynamic deflections may result from this type of loading and further these deflections may be considerably influenced by the system of initial stress.

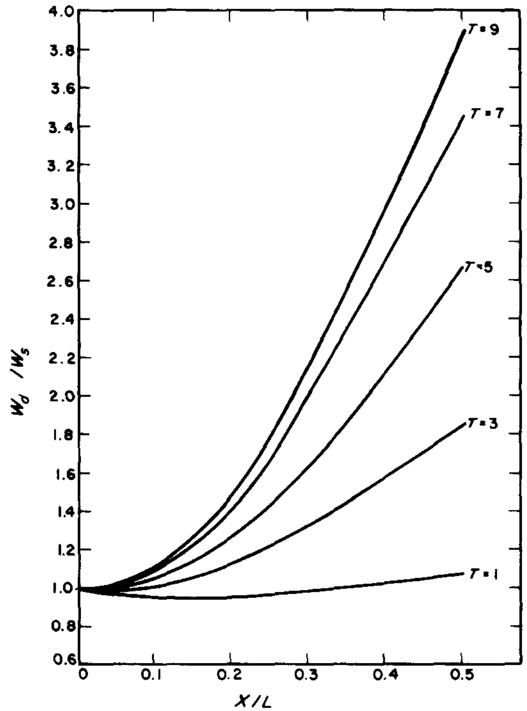

FIG. 4. Ratio of dynamic deflection to static deflection vs load movement amplitude for shells with $l/a = 4{\cdot}0$, $h/a = 0{\cdot}01$, $v = 0{\cdot}3$, $x_1 = x = l/2$, $\beta = 0$, $\alpha_1 = \alpha_2 = 0$.

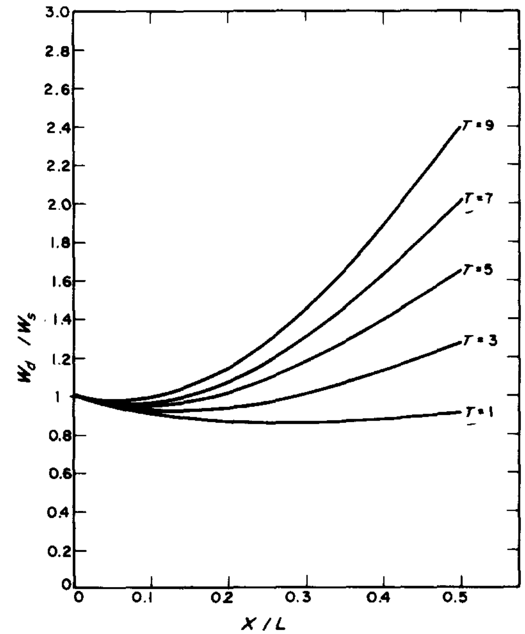

FIG. 5. Ratio of dynamic deflection to static deflection vs load movement amplitude for shells with $l/a = 4{\cdot}0$, $h/a = 0{\cdot}01$, $v = 0{\cdot}3$, $x_1 = x = l/2$, $\beta = 0$, $\alpha_1 = 0{\cdot}5 \times 10^{-3}$, $\alpha_2 = 0{\cdot}1 \times 10^{-3}$.

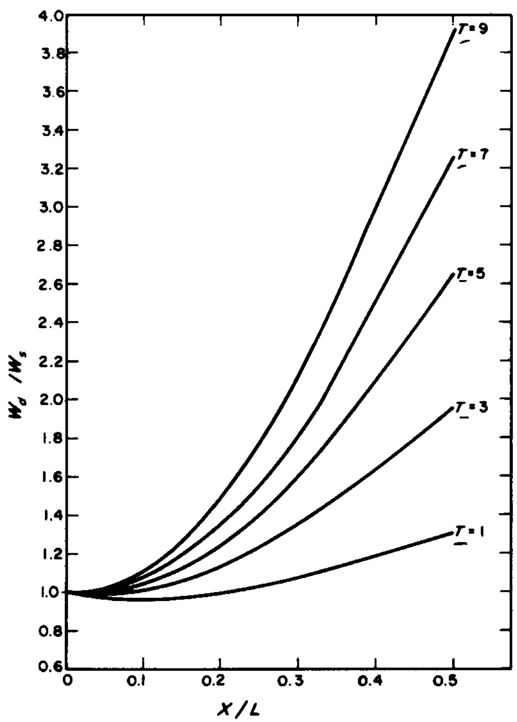

FIG. 6. Ratio of dynamic deflection to static deflection vs load movement amplitude for shells with $l/a = 4{\cdot}0$, $h/a = 0{\cdot}01$, $v = 0{\cdot}3$, $x_1 = x = l/2$, $\beta = 0$, $\alpha_1 = -0{\cdot}5 \times 10^{-4}$, $\alpha_2 = -0{\cdot}2 \times 10^{-4}$.

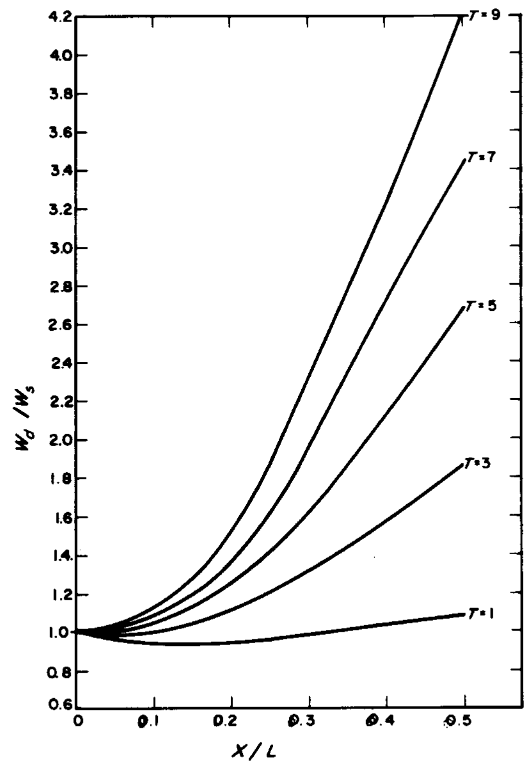

FIG. 7. Ratio of dynamic deflection to static deflection vs load movement amplitude for shells with $l/a = 6{\cdot}0$, $h/a = 0{\cdot}01$, $v = 0{\cdot}3$, $x_1 = x = l/2$, $\beta = 0$, $\alpha_1 = \alpha_2 = 0$.

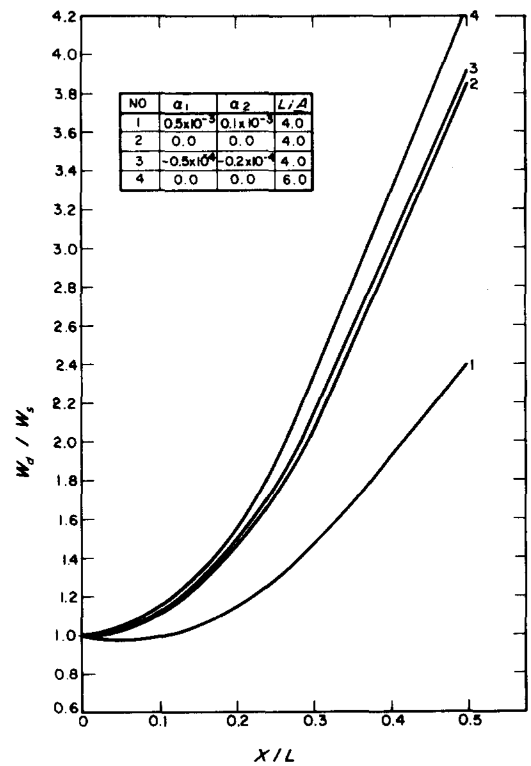

FIG. 8. Ratio of dynamic deflection to static deflection vs load movement amplitude at dimensionless time $T = 9$.

Figure 7 illustrates the effect of increasing the L/a ratio to 6·0 with a system of no initial stress. Finally Fig. 8 provides a basis of comparison of Figs. 4–7 for a dimensionless period of time $T = 9$.

6. CONCLUSIONS

The general solution to the dynamic problem of pressurized or unpressurized simply supported cylindrical shells subjected to arbitrary time-dependent surface forces has been derived in the integral representation from the standpoint of retaining radial, axial and circumferential inertia forces, and the associated Green's tensor is also presented explicitly in terms of modal functions.

This paper has demonstrated the direct application of the integral solution to specific problems of a sinusoidally oscillating ring load and a cyclic travelling concentrated force to determine analytically the transient and steady state response. It is found that in both problems there are, theoretically, infinite opportunities of resonance for each mode of vibration which can be strongly excited by axial movement frequency $\gamma = \Omega$, $\Omega/2$, $\Omega/3, \ldots$.

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APPENDIX 1

Although the analysis is presented based on the Flugge's equations, it is interesting to note that the general solution together with the Green's tensor representation and the analytical solutions obtained for the specific problems are also valid for the Donnell's equations if proper changes of definitions are made as follows:

$$A_{11} = \frac{\partial^2}{\partial\alpha^2}+\left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\beta^2}, \qquad A_{12} = \left(\frac{1+\nu}{2}\right)\frac{\partial^2}{\partial\alpha\,\partial\beta} = A_{21}$$

$$A_{13} = \nu\frac{\partial}{\partial\alpha} = A_{31}, \qquad A_{22} = \left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}, \qquad A_{23} = \frac{\partial}{\partial\beta} = A_{32} \tag{A–1}$$

$$A_{33} = 1+k\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)^2-\alpha_1\frac{\partial^2}{\partial\alpha^2}-\alpha_2\frac{\partial^2}{\partial\beta^2}$$

$$
\begin{aligned}
&A_0 = \left[\left(\frac{1-\nu}{2}\right)n^2\right], \qquad A_1 = \left[\lambda^2+\left(\frac{1-\nu}{2}\right)n^2\right], \qquad A_2 = \left[\left(\frac{1+\nu}{2}\right)\lambda n\right] \\
&A_3 = \nu\lambda, \qquad A_4 = \lambda^2, \qquad A_5 = \nu\lambda \\
&B_0 = \left(\frac{1-\nu}{2}\right)\lambda^2, \qquad B_2 = n^2+\left(\frac{1-\nu}{2}\right)\lambda^2, \qquad B_3 = n \\
&C_3 = [1+k(\lambda^2+n^2)^2+\alpha_1\lambda^2+\alpha_2 n^2], \qquad C_4 = [1+k\lambda^4+\alpha_1\lambda^2].
\end{aligned} \tag{A–2}
$$

By replacing equations (3) and (16) by equations (A–1) and (A–2) respectively, the argument and result presented in the foregoing analysis are applicable to the Donnell's equations equally well.

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Абстракт—Выводится интегральное решение для вектора перемещений серединной поверхности конечной, свободно опертой, тонкой, круглой цилиндрической оболочки, с начальными двухосными напряжениями, подверженной действию произвольных, трехраэмерных, зависящих от времени, поверхностных нагрузок. Решение дается в виде ядра присоединенного тензора Грина, основааного на уравнениях флюгге. В случае непосредственных применений, работа дает теоретический анализ для: (а) динамического поведения предварительно напряженной цилиндрической оболочки вследствие циклической, подвижной, осесимметрической, кольцевой нагрузки и (б) нестационарного и стационарного поведения предварительно напряженной оболочки, подверженной действию циклической, движущейся сосредоточенной силы. Оказывается что в этих двух задачах теоретически существует бесконечное число частот движения нагрузки для того, чтобы возбуждать близи резонанса каждий вид колебаний оболочки, либо с начальными напряжениями, либо без. Общее динамическое решение предварительно напряженных оболочек вместе с изображением тензора Грина и аналитические решения особенных задач являются, также, важными для уравнений Донелла, если только определение понятия некоторых коэффициентов правильно установлено.